Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (mm 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

Linear Diophantine Sheffer stroke ideals in Sheffer stroke Hilbert algebras

Young Bae Jun

Received 26 August 2025; Revised 22 September 2025; Accepted 30 September 2025

Abstract. The purpose of this paper is to introduce the concept of linear Diophantine Sheffer stroke ideal in Sheffer stroke Hilbert algebra, and to investigate several properties. We will address the characterization of linear Diophantine Sheffer stroke ideal, and explore the conditions for the linear Diophantine set to be linear Diophantine Sheffer stroke ideal. After constructing subsets ε_t, δ^s, $\varepsilon_{q(t)}$, $\delta^{q(s)}$, $(\hbar, \in)_t$, $(\eth, \in)^t$, $(\hbar, q)_t$, and $(\eth, q)^t$, we will explore the conditions under which they become the ideal of Sheffer stroke Hilbert algebra.

2020 AMS Classification: 03B05, 03G25, 06F35, 08A72

Keywords: Sheffer stroke Hilbert algebra, Diophantine structure, linear Diophantine set, linear Diophantine Sheffer stroke ideal.

Corresponding Author: Y. B. Jun (skywine@gmail.com)

1. Introduction

The Sheffer stroke "|", which is introduced by Sheffer [1], gives a minimal algebraic signature for Boolean algebras in algebraic presentations, and all propositional tautologies can axiomatized using only formulas built from |. Logical connectives such as $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, etc. can be defined using the Sheffer stroke "|" which is a universal connective. For example, $\neg p \equiv p|p$ (negation), $p \rightarrow q \equiv p|(q|q)$ (implication), $p \wedge q \equiv (p|q)|(p|q)$ (conjunction), $1 \equiv p|(p|p)$ (truth constant), $p \vee q \equiv ((p|p)|(q|q))|((p|p)|(q|q))$ (disjunction), and the biconditional $p \leftrightarrow q$ by

$$p \leftrightarrow q \equiv ((p|(q|q))|(q|(p|p))) \,|\, ((p|(q|q))|(q|(p|p))) .$$

A Hilbert algebra is usually defined with an implication $\rightarrow$ and a constant 1. But since $\rightarrow$ and 1 can both be defined in terms of the Sheffer stroke "|", Oner et al. [2] introduced the Sheffer stroke Hilbert algebras, and described its deductive systems and ideals. Katican and Bordbar [3] provided a new characterization of Sheffer stroke

Hilbert algebras by suggesting ideals and stabilizers. Jun and Oner [4] addressed weak filters and multipliers in Sheffer stroke Hilbert algebras. Also, Jun, Oner, Katican and Borumand Saeid studied fuzzy ideals of Sheffer stroke Hilbert algebras (See [5, 6]).

A linear Diophantine equation is a type of Diophantine equation that is linear in the unknowns, and it is an equation of the form

$$\mathfrak{a}x + \mathfrak{b}y = \mathfrak{c}$$

where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are given integers, and the unknowns x, y are sought as integers. Using the idea of linear Diophantine equation, Riaz and Hashmi [7] introduced the concept of a linear Diophantine fuzzy set in a set X by using two fuzzy sets f and g in X and two reference parameters α and β in $[0, 1]$, and it satisfies the condition

$$0 \leq \alpha f(x) + \beta g(x) \leq 1 \tag{1.1}$$

for all $x \in X$ with $0 \leq \alpha + \beta \leq 1$. However, in general, the condition (1.1) is always true for all α and β with $0 \leq \alpha + \beta \leq 1$. This means that the condition (1.1) is not in a vibrant state. Therefore, we feel the need to reassign the roles of two reference parameters α and β in order to activate the condition (1.1). Based on this intention, we invigorate the condition (1.1) by introducing a linear Diophantine set, and apply it to Sheffer stroke Hilbert algebras. We introduce the concept of a linear Diophantine Sheffer stroke ideal in Sheffer stroke Hilbert algebras, and investigate several properties. We address the characterization of the linear Diophantine Sheffer stroke ideal. We explore the conditions under which a linear Diophantine set becomes a linear Diophantine Sheffer stroke ideal. We construct subsets ε_t, δ^s, $\varepsilon_{q(t)}$, $\delta^{q(s)}$, $(\hbar, \in)_t$, $(\eth, \in)^t$, $(\hbar, q)_t$, and $(\eth, q)^t$, and so on to explore the conditions under which they become the ideal of a Sheffer stroke Hilbert algebra.

2. Preliminaries

Definition 2.1 ([1])**.** Let $(X, |)$ be a groupoid. Then the operation $|$ is said to be *Sheffer stroke* or *Sheffer operation*, if it satisfies

- (s1) $\mathfrak{a}|\mathfrak{b} = \mathfrak{b}|\mathfrak{a}$,
- (s2) $(\mathfrak{a}|\mathfrak{a})|(\mathfrak{a}|\mathfrak{b}) = \mathfrak{a}$,
- (s3) $\mathfrak{a}|((\mathfrak{b}|\mathfrak{c})|(\mathfrak{b}|\mathfrak{c})) = ((\mathfrak{a}|\mathfrak{b})|(\mathfrak{a}|\mathfrak{b}))|\mathfrak{c}$,
- (s4) $(\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b})))|(\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b}))) = \mathfrak{a}$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

Definition 2.2 ([2])**.** A *Sheffer stroke Hilbert algebra* is a groupoid $(X, |)$ with a Sheffer stroke $|$ that satisfies

- (sH1) $(\mathfrak{a}|((A)|(A)))|(((B)|((C)|(C)))|((B)|((C)|(C)))) = \mathfrak{a}|(\mathfrak{a}|\mathfrak{a})$, where $A := \mathfrak{b}|(\mathfrak{c}|\mathfrak{c})$, $B := \mathfrak{a}|(\mathfrak{b}|\mathfrak{b})$ and $C := \mathfrak{a}|(\mathfrak{c}|\mathfrak{c})$,
- (sH2) $\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) = \mathfrak{b}|(\mathfrak{a}|\mathfrak{a}) = \mathfrak{a}|(\mathfrak{a}|\mathfrak{a}) \ \Rightarrow \ \mathfrak{a} = \mathfrak{b}$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

Every Sheffer stroke Hilbert algebra $(X, |)$ satisfies $\mathfrak{a}|(\mathfrak{a}|\mathfrak{a}) = \mathfrak{b}|(\mathfrak{b}|\mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \in X$ (See [2, Lemma 3.4]) and it is denoted by 1. Then $(X, |)$ contains the algebraic constant 1. Because of this background, the Sheffer stroke Hilbert algebra $(X, |)$ can be expressed as $\mathcal{X} := (X, |, 1)$.

Let $\mathcal{X} := (X, |, 1)$ be a Sheffer stroke Hilbert algebra. Then the order relation $\leq_X$ on X is defined as follows:

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \leq_X \mathfrak{b} \;\Leftrightarrow\; \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) = 1). \tag{2.1}$$

We observe that the relation $\leq_X$ is a partial order in a Sheffer stroke Hilbert algebra $\mathcal{X} := (X, |, 1)$ (See [2]) and 1 is the largest element of X.

Proposition 2.3 ([2]). *Every Sheffer stroke Hilbert algebra $\mathcal{X} := (X, |, 1)$ satisfies*

$$\mathfrak{a}|(\mathfrak{a}|\mathfrak{a}) = 1, \tag{2.2}$$

$$\mathfrak{a}|(1|1) = 1, \tag{2.3}$$

$$((\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}) = \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}), \tag{2.4}$$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

Proposition 2.4 ([2]). *If a Sheffer stroke Hilbert $\mathcal{X} := (X, |, 1)$ contains the least element 0, then the following assertions are valid.*

$$0|0 = 1,\ 1|1 = 0, \tag{2.5}$$

$$\mathfrak{a}|1 = \mathfrak{a}|\mathfrak{a},\ \mathfrak{a}|0 = 1. \tag{2.6}$$

Definition 2.5 ([2]). Let $\mathcal{X} := (X, |, 1)$ be a Sheffer stroke Hilbert algebra with the smallest element 0. A subset A of X is called an *ideal* of $\mathcal{X} := (X, |, 1)$, if it satisfies

$$0 \in A, \tag{2.7}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{b} \in A,\ (\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) \in A \;\Rightarrow\; \mathfrak{a} \in A). \tag{2.8}$$

3. Linear Diophantine Sheffer stroke ideals

Using the idea of linear Diophantine equation, Riaz and Hashmi [7] introduced the concept of a linear Diophantine fuzzy set in X as follows.

Definition 3.1 ([7]). Let X be a nonempty reference set. A *linear Diophantine fuzzy set* on X is an object of the form:

$$\mathcal{L} = \{(x; (\hbar(x), \eth(x)), (\varepsilon, \delta)) \mid x \in X\},$$

where $\hbar(x)$, $\eth(x)$, ε, $\delta \in (0, 1]$ are membership, non-membership and reference parameters respectively. These grades satisfy the condition

$$0 \leq \varepsilon\hbar(x) + \delta\eth(x) \leq 1 \text{ for all } x \in X \text{ with } 0 \leq \varepsilon + \delta \leq 1. \tag{3.1}$$

In the linear Diophantine fuzzy set on X, since $\hbar(x)$, $\eth(x)$, ε, and δ are within $(0, 1]$, we know that if $0 \leq \varepsilon + \delta \leq 1$, then the condition $0 \leq \varepsilon\hbar(x) + \delta\eth(x) \leq 1$ is always true for all $x \in X$. This raises concerns about the new role of reference parameters.

Let X be a nonempty reference set, $\varepsilon, \delta \in (0, 1]$ be reference parameters, and $\hbar$ and $\eth$ be functions from X to $[0, 1]$.

Consider an object of the form:

$$\mathcal{L} = \{(x; (\hbar(x), \eth(x)), (\varepsilon, \delta)) \mid x \in X\}, \tag{3.2}$$

and it is simply denoted by $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$, and we say it is a *Diophantine structure* in X. Since reference parameters ε and δ are freely determined for each element $x \in X$, the reference parameters ε and δ given to the point $x \in X$ in the future will be expressed as ε_x and δ_x respectively, and they are considered to be in $(0, 1]$ since $\varepsilon, \delta \in (0, 1]$.

Definition 3.2. A Diophantine structure $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ in X is called a *linear Diophantine set* in X, if it satisfies $0 < \varepsilon_x + \delta_x \leq 1$ and

$$0 < \tfrac{1}{\varepsilon_x}\hbar(x) + \tfrac{1}{\delta_x}\eth(x) \leq 1 \tag{3.3}$$

for all $x \in X$.

Example 3.3. Let $X = \{\diamond_1, \diamond_2, \diamond_3, \diamond_4\}$ be a reference set and $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ be a Diophantine structure in X given by the two columns to the left in Table 1.

Table 1. Tabular representation of $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$

$x \in X$	$\mathcal{L}(x) := (\hbar(x), \eth(x); \varepsilon_x, \delta_x)$	$\frac{1}{\varepsilon_x}\hbar(x) + \frac{1}{\delta_x}\eth(x)$	$\varepsilon_x + \delta_x$
$\diamond_1$	$(0.01, 0.02; 0.50, 0.49)$	≈ 0.0608	0.99
$\diamond_2$	$(0.12, 0.13; 0.48, 0.49)$	≈ 0.5153	0.97
$\diamond_3$	$(0.06, 0.07; 0.61, 0.37)$	≈ 0.2876	0.98
$\diamond_4$	$(0.17, 0.23; 0.54, 0.37)$	≈ 0.9364	0.91

Then the two columns to the right of Table 1 verify that $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ is a linear Diophantine set in X.

In what follows, let $\mathcal{X} := (X, |, 1)$ (or simply $\mathcal{X}$) denote a Sheffer stroke Hilbert algebra with the smallest element 0 unless otherwise specified. Also, we use the symbol x^y instead of $x|(y|y)$, and the element (x, x, x) of $X^3 = X \times X \times X$ will be denoted by $[x]^3$ for all $x, y \in X$.

In a linear Diophantine set $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ in X, we use the next sets:

- $\varepsilon(\min) := \{(x, y, z) \in X^3 \mid \frac{1}{\varepsilon_x} \leq \frac{1}{\min\{\varepsilon_y, \varepsilon_z\}}\}$,
- $\delta(\max) := \{(x, y, z) \in X^3 \mid \frac{1}{\delta_x} \geq \frac{1}{\max\{\delta_y, \delta_z\}}\}$,
- $(\hbar, \in)_t := \{x \in X \mid \hbar(x) \geq t\}$,
- $(\eth, \in)^s := \{x \in X \mid \eth(x) \leq s\}$,
- $(\hbar, q)_t := \{x \in X \mid \hbar(x) + t > 1\}$,
- $(\eth, q)^s := \{x \in X \mid \eth(x) + s < 1\}$,

where t and s are within $(0, 1)$. We note that $(\hbar, q)_0 = \varnothing = (\eth, q)^1$.

Definition 3.4. A linear Diophantine set $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ in X is called a *linear Diophantine Sheffer stroke ideal* of $\mathcal{X}$, if it satisfies

$$\begin{array}{l} (\forall x, \mathfrak{a}, z, \mathfrak{c} \in X) \\ \left(\begin{array}{l} ((0, x, x), (0, \mathfrak{a}, \mathfrak{a})) \in \varepsilon(\min) \times \delta(\max) \\ ((z, x, z^x|z^x), (\mathfrak{c}, \mathfrak{a}, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})) \in \varepsilon(\min) \times \delta(\max) \end{array} \right), \end{array} \tag{3.4}$$

$$(\forall t, s \in (0,1)) \left((\hbar, \in)_t \neq \varnothing \neq (\eth, \in)^s \ \vdash (0,0) \in (\hbar, \in)_t \times (\eth, \in)^s \right), \tag{3.5}$$

and

$$(\forall x, \mathfrak{a}, z, \mathfrak{c} \in X)(\forall t_1, t_2, s_1, s_2 \in (0,1)) \left(\begin{array}{l} \left(\begin{array}{l} (x, \mathfrak{a}) \in (\hbar, \in)_{t_1} \times (\eth, \in)^{s_1} \\ (z^x | z^x, \mathfrak{c}^{\mathfrak{a}} | \mathfrak{c}^{\mathfrak{a}}) \in (\hbar, \in)_{t_2} \times (\eth, \in)^{s_2} \end{array} \right) \\ \vdash (z, \mathfrak{c}) \in (\hbar, \in)_{\min\{t_1, t_2\}} \times (\eth, \in)^{\max\{s_1, s_2\}} \end{array} \right). \tag{3.6}$$

Example 3.5. Consider a set $X = \{\diamond_1, \diamond_2, \diamond_3, \diamond_0\}$, and define a Sheffer stroke $|$ by Table 2.

Table 2. Cayley table for the Sheffer stroke $|$

$\vert$	$\diamond_1$	$\diamond_2$	$\diamond_3$	$\diamond_0$
$\diamond_1$	$\diamond_0$	$\diamond_3$	$\diamond_2$	$\diamond_1$
$\diamond_2$	$\diamond_3$	$\diamond_3$	$\diamond_1$	$\diamond_1$
$\diamond_3$	$\diamond_2$	$\diamond_1$	$\diamond_2$	$\diamond_1$
$\diamond_0$	$\diamond_1$	$\diamond_1$	$\diamond_1$	$\diamond_1$

Then $\mathcal{X} := (X, |, \diamond_1)$ is a Sheffer stroke Hilbert algebra with the smallest element $\diamond_0$ (See [2]). Let $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ be a linear Diophantine set in X given by left two columns in Table 3.

Table 3. Tabular representation of $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$

$x \in X$	$\mathcal{L}(x) := (\hbar(x), \eth(x); \varepsilon_x, \delta_x)$	$\frac{1}{\varepsilon_x}\hbar(x) + \frac{1}{\delta_x}\eth(x)$	$\varepsilon_x + \delta_x$
$\diamond_1$	$(0.15, 0.12; 0.63, 0.21)$	≈ 0.8095	0.84
$\diamond_2$	$(0.15, 0.11; 0.66, 0.21)$	≈ 0.7511	0.87
$\diamond_3$	$(0.23, 0.12; 0.63, 0.19)$	≈ 0.9967	0.62
$\diamond_0$	$(0.25, 0.09; 0.79, 0.17)$	≈ 0.8459	0.96

Then the two columns to the right of Table 3 verify that $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ is a linear Diophantine set in X. By simple calculations, we know that $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ is a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$.

Theorem 3.6. *A linear Diophantine set $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ in X is a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$ if and only if it satisfies*

$$(\forall x, \mathfrak{a} \in X) \left(\begin{array}{l} \varepsilon_0 \geq \varepsilon_x, \ \delta_0 \leq \delta_{\mathfrak{a}} \\ \varepsilon_z \geq \min\{\varepsilon_x, \varepsilon_{z^x|z^x}\} \\ \delta_{\mathfrak{c}} \leq \max\{\delta_{\mathfrak{a}}, \delta_{\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}}\} \end{array} \right), \tag{3.7}$$

$$(\forall x, \mathfrak{a} \in X) \left(\ \hbar(0) \geq \hbar(x), \ \eth(0) \leq \eth(\mathfrak{a}) \ \right), \tag{3.8}$$

and

$$(\forall x, \mathfrak{a}, z, \mathfrak{c} \in X) \left(\begin{array}{l} \hbar(z) \geq \min\{\hbar(x), \hbar(z^x | z^x)\} \\ \eth(\mathfrak{c}) \leq \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}} | \mathfrak{c}^{\mathfrak{a}})\} \end{array} \right). \tag{3.9}$$

Proof. Assume that the linear Diophantine set $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ is a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$. If $\varepsilon_0 < \varepsilon_x$ and $\delta_0 > \delta_{\mathfrak{a}}$ for some $x, \mathfrak{a} \in X$, then $\frac{1}{\varepsilon_0} > \frac{1}{\varepsilon_x} = \frac{1}{\min\{\varepsilon_x, \varepsilon_x\}}$ and $\frac{1}{\delta_0} < \frac{1}{\delta_{\mathfrak{a}}} = \frac{1}{\max\{\delta_{\mathfrak{a}}, \delta_{\mathfrak{a}}\}}$. Thus

$$((0, x, x), (0, \mathfrak{a}, \mathfrak{a})) \notin \varepsilon(\min) \times \delta(\max),$$

a contradiction. So $\varepsilon_0 \geq \varepsilon_x$ and $\delta_0 \leq \delta_{\mathfrak{a}}$ for all $x, \mathfrak{a} \in X$.

Suppose that there exist $x, \mathfrak{a}, z, \mathfrak{c} \in X$ such that $\varepsilon_z < \min\{\varepsilon_x, \varepsilon_{z^x|z^x}\}$ and $\delta_{\mathfrak{c}} > \max\{\delta_{\mathfrak{a}}, \delta_{\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}}\}$. Then $\frac{1}{\varepsilon_z} > \frac{1}{\min\{\varepsilon_x, \varepsilon_{z^x|z^x}\}}$ and $\frac{1}{\delta_{\mathfrak{c}}} < \frac{1}{\max\{\delta_{\mathfrak{a}}, \delta_{\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}}\}}$. Thus

$$((z, x, z^x|z^x), (\mathfrak{c}, \mathfrak{a}, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})) \notin \varepsilon(\min) \times \delta(\max)$$

which is a contradiction. So $\varepsilon_z \geq \min\{\varepsilon_x, \varepsilon_{z^x|z^x}\}$ and $\delta_{\mathfrak{c}} \leq \max\{\delta_{\mathfrak{a}}, \delta_{\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}}\}$ for all $x, \mathfrak{a}, z, \mathfrak{c} \in X$. Since $(x, \mathfrak{a}) \in (\hbar, \in)_{\hbar(x)} \times (\eth, \in)^{\eth(\mathfrak{a})}$ for every $x, \mathfrak{a} \in X$, we know that $(\hbar, \in)_{\hbar(x)} \neq \varnothing \neq (\eth, \in)^{\eth(\mathfrak{a})}$. Hence

$$(0, 0) \in (\hbar, \in)_{\hbar(x)} \times (\eth, \in)^{\eth(\mathfrak{a})}$$

by (3.5), which shows that $\hbar(0) \geq \hbar(x)$ and $\eth(0) \leq \eth(\mathfrak{a})$, i.e., (3.8) is valid.

Let $x, \mathfrak{a}, z, \mathfrak{c} \in X$ be such that $t_1 := \hbar(x)$, $t_2 := \hbar(z^x|z^x)$, $s_1 := \eth(\mathfrak{a})$, and $s_2 := \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})$. Then $(x, \mathfrak{a}) \in (\hbar, \in)_{t_1} \times (\eth, \in)^{s_1}$ and $(z^x|z^x, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \in (\hbar, \in)_{t_2} \times (\eth, \in)^{s_2}$. Thus by (3.6),

$$(z, \mathfrak{c}) \in (\hbar, \in)_{\min\{t_1, t_2\}} \times (\eth, \in)^{\max\{s_1, s_2\}}.$$

So $\hbar(z) \geq \min\{t_1, t_2\} = \min\{\hbar(x), \hbar(z^x|z^x)\}$ and

$$\eth(\mathfrak{c}) \leq \max\{s_1, s_2\} = \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})\}.$$

Conversely, let $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ be a linear Diophantine set in X sarisfying the conditions (3.7), (3.8) and (3.9). Using (3.7), we have $\frac{1}{\varepsilon_0} \leq \frac{1}{\varepsilon_x} = \frac{1}{\min\{\varepsilon_x, \varepsilon_x\}}$, $\frac{1}{\delta_0} \geq \frac{1}{\delta_{\mathfrak{a}}} = \frac{1}{\max\{\delta_{\mathfrak{a}}, \delta_{\mathfrak{a}}\}}$, $\frac{1}{\varepsilon_z} \leq \frac{1}{\min\{\varepsilon_x, \varepsilon_{z^x|z^x}\}}$ and $\frac{1}{\delta_{\mathfrak{c}}} \geq \frac{1}{\max\{\delta_{\mathfrak{a}}, \delta_{\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}}\}}$. Then $((0, x, x), (0, \mathfrak{a}, \mathfrak{a})) \in \varepsilon(\min) \times \delta(\max)$ and $((z, x, z^x|z^x), (\mathfrak{c}, \mathfrak{a}, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})) \in \varepsilon(\min) \times \delta(\max)$. Let $t, s \in (0, 1)$ be such that $(\hbar, \in)_t \neq \varnothing \neq (\eth, \in)^s$, say $(x, \mathfrak{a}) \in (\hbar, \in)_t \times (\eth, \in)^s$. Then $\hbar(0) \geq \hbar(x) \geq t$ and $\eth(0) \leq \eth(\mathfrak{a}) \leq s$ by (3.8), which means that $(0, 0) \in (\hbar, \in)_t \times (\eth, \in)^s$. Let $x, \mathfrak{a}, z, \mathfrak{c} \in X$ and $t_1, t_2, s_1, s_2 \in (0, 1)$ be such that $(x, \mathfrak{a}) \in (\hbar, \in)_{t_1} \times (\eth, \in)^{s_1}$ and

$$(z^x|z^x, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \in (\hbar, \in)_{t_2} \times (\eth, \in)^{s_2}.$$

Using (3.9), we get $\hbar(z) \geq \min\{\hbar(x), \hbar(z^x|z^x)\} \geq \min\{t_1, t_2\}$ and

$$\eth(\mathfrak{c}) \leq \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})\} \leq \max\{s_1, s_2\},$$

that is, $(z, \mathfrak{c}) \in (\hbar, \in)_{\min\{t_1, t_2\}} \times (\eth, \in)^{\max\{s_1, s_2\}}$. Thus $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ is a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$. □

Proposition 3.7. *Every linear Diophantine Sheffer stroke ideal $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ of $\mathcal{X}$ satisfies*

$$(\forall x, \mathfrak{a}, y, \mathfrak{b} \in X) \left(\begin{array}{l} x \leq_X y, \mathfrak{a} \leq_X \mathfrak{b} \\ \vdash ((x, y, 0), (\mathfrak{a}, \mathfrak{b}, 0)) \in \varepsilon(\min) \times \delta(\max) \end{array} \right), \tag{3.10}$$

$$(3.11)\qquad \begin{array}{c}(\forall x,\mathfrak{a},y,\mathfrak{b}\in X)(\forall t,s\in(0,1))\\ \left(\begin{array}{l}\left(\begin{array}{l}x\leq_X y,\ \mathfrak{a}\leq_X \mathfrak{b}\\ (y,\mathfrak{b})\in(\hbar,\in)_t\times(\eth,\in)^s\end{array}\right)\\ \vdash (x,\mathfrak{a})\in(\hbar,\in)_t\times(\eth,\in)^s\end{array}\right).\end{array}$$

and

$$(3.12)\qquad \begin{array}{l}(\forall x,\mathfrak{a},y,\mathfrak{b}\in X)(\forall t,s\in(0,1))\\ \left(\begin{array}{l}(x,\mathfrak{a})\in(\hbar,\in)_{t_1}\times(\eth,\in)^{s_1},\ (y,\mathfrak{b})\in(\hbar,\in)_{t_2}\times(\eth,\in)^{s_2}\\ \vdash (x^y|(y|y),\mathfrak{a}^{\mathfrak{b}}|(\mathfrak{b}|\mathfrak{b}))\in(\hbar,\in)_{\min\{t_1,t_2\}}\times(\eth,\in)^{\max\{s_1,s_2\}}\end{array}\right).\end{array}$$

Proof. Let $x,\mathfrak{a},y,\mathfrak{b}\in X$ and $t,s\in(0,1)$ be such that $x\leq_X y$, $\mathfrak{a}\leq_X\mathfrak{b}$ and $(y,\mathfrak{b})\in(\hbar,\in)_t\times(\eth,\in)^s$. Then $x^y=1=\mathfrak{a}^{\mathfrak{b}}$. Thus

$$\begin{aligned}((x,y,0),(\mathfrak{a},\mathfrak{b},0))&=((x,y,1|1),(\mathfrak{a},\mathfrak{b},1|1))\\&=((x,y,x^y|x^y),(\mathfrak{a},\mathfrak{b},\mathfrak{a}^{\mathfrak{b}}|\mathfrak{a}^{\mathfrak{b}}))\\&\in\varepsilon(\min)\times\delta(\max)\end{aligned}$$

and $(x^y|x^y,\mathfrak{a}^{\mathfrak{b}}|\mathfrak{a}^{\mathfrak{b}})=(1|1,1|1)=(0,0)\in(\hbar,\in)_t\times(\eth,\in)^s$ by (2.5), (3.4) and (3.5). So $(x,\mathfrak{a})\in(\hbar,\in)_t\times(\eth,\in)^s$ by (3.6). Let $(x,\mathfrak{a})\in(\hbar,\in)_{t_1}\times(\eth,\in)^{s_1}$ and $(y,\mathfrak{b})\in(\hbar,\in)_{t_2}\times(\eth,\in)^{s_2}$ for all $x,\mathfrak{a},y,\mathfrak{b}\in X$. For every $x,y\in X$, if we put

$$g_x^y:=((x^y|(y|y))|(y|y))|((x^y|(y|y))|(y|y)),$$

then

$$\begin{aligned}f_x^y&:=(g_x^y|(x|x))|(g_x^y|(x|x))\\&=((((x^y|(y|y))|(y|y))|((x^y|(y|y))|(y|y)))|(x|x))|\\&\qquad((((x^y|(y|y))|(y|y))|((x^y|(y|y))|(y|y)))|(x|x))\\&=((x^y|x^y)|(x|x))|((x^y|x^y)|(x|x))\\&=((y|y)|(x^x|x^x))|((y|y)|(x^x|x^x))\\&=((y|y)|(1|1))|((y|y)|(1|1))\\&=1|1=0\end{aligned}$$

by (s1), (s3), (2.2), (2.3), (2.4) and (2.5). It follows from (3.5) that

$$\begin{aligned}&((g_x^y|(x|x))|(g_x^y|(x|x)),(g_{\mathfrak{a}}^{\mathfrak{b}}|(\mathfrak{a}|\mathfrak{a}))|(g_{\mathfrak{a}}^{\mathfrak{b}}|(\mathfrak{a}|\mathfrak{a})))\\&=(f_x^y,f_{\mathfrak{a}}^{\mathfrak{b}})=(0,0)\in(\hbar,\in)_{t_2}\times(\eth,\in)^{s_2}.\end{aligned}$$

Using the condition (3.6) leads to

$$\begin{aligned}&\big(((x^y|(y|y))|(y|y))|((x^y|(y|y))|(y|y)),\\&\qquad((\mathfrak{a}^{\mathfrak{b}}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}))|((\mathfrak{a}^{\mathfrak{b}}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}))\big)\\&=(g_x^y,g_{\mathfrak{a}}^{\mathfrak{b}})\in(\hbar,\in)_{\min\{t_1,t_2\}}\times(\eth,\in)^{\max\{s_1,s_2\}}.\end{aligned}$$

Since $(y,\mathfrak{b})\in(\hbar,\in)_{t_2}\times(\eth,\in)^{s_2}$, it follows from (3.6) that

$$(x^y|(y|y),\mathfrak{a}^{\mathfrak{b}}|(\mathfrak{b}|\mathfrak{b}))\in(\hbar,\in)_{\min\{t_1,t_2\}}\times(\eth,\in)^{\max\{s_1,s_2\}}.$$

This completes the proof. □

Now we consider the conditions under which a linear Diophantine set $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ in X can be a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$.

Lemma 3.8 ([2])**.** *In a Sheffer stroke Hilbert algebra $\mathcal{X} := (X, |, 1)$, the set $\{x, z\}$ has the the least upper bound $z^x|(x|x)$ for every $x, z \in X$.*

Theorem 3.9. *Let $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ be a linear Diophantine set in X satisfying (3.4). If it satisfies the conditions (3.11) and (3.12), then it is a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$.*

Proof. Let $t, s \in (0, 1)$ be such that $(\hbar, \in)_t \neq \varnothing \neq (\eth, \in)^s$, say $(y, \mathfrak{b}) \in (\hbar, \in)_t \times (\eth, \in)^s$. Since 0 is the smallest element in $\mathcal{X}$, we have $(0, 0) \in (\hbar, \in)_t \times (\eth, \in)^s$ by (3.11). Let $x, \mathfrak{a}, z, \mathfrak{c} \in X$ and $t_1, t_2, s_1, s_2 \in (0, 1)$ be such that

$$(x, \mathfrak{a}) \in (\hbar, \in)_{t_2} \times (\eth, \in)^{s_2} \text{ and } (z^x|z^x, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \in (\hbar, \in)_{t_1} \times (\eth, \in)^{s_1}.$$

Then

$$\begin{aligned}
&((z^x|(x|x)), (\mathfrak{c}^{\mathfrak{a}}|(\mathfrak{a}|\mathfrak{a})))\\
&= \big((z|(((x|x)|(x|x))|((x|x)|(x|x))))|(x|x),\\
&\qquad (\mathfrak{c}|(((\mathfrak{a}|\mathfrak{a})|(\mathfrak{a}|\mathfrak{a}))|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{a}|\mathfrak{a}))))|(\mathfrak{a}|\mathfrak{a})\big)\\
&= (((z^x|z^x)|(x|x))|(x|x), ((\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})|(\mathfrak{a}|\mathfrak{a}))|(\mathfrak{a}|\mathfrak{a}))\\
&\in (\hbar, \in)_{\min\{t_1,t_2\}} \times (\eth, \in)^{\max\{s_1,s_2\}}
\end{aligned}$$

by (s2), (s3) and (3.12). The Lemma 3.8 induces $z \leq_X z^x|(x|x)$ and $\mathfrak{c} \leq_X \mathfrak{c}^{\mathfrak{a}}|(\mathfrak{a}|\mathfrak{a})$. It follows from (3.11) that

$$(z, \mathfrak{c}) \in (\hbar, \in)_{\min\{t_1,t_2\}} \times (\eth, \in)^{\max\{s_1,s_2\}}.$$

Thus $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ is a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$. □

Lemma 3.10. *In a linear Diophantine set $\mathcal{L} := (\hbar, \eth; \varepsilon, \delta)$ in X, consider the following assertions:*

$$(\hbar, \in)_t \neq \varnothing \neq (\eth, \in)^s \ \vdash \left(\begin{array}{l} (0, 0) \in (\hbar, \in)_t \times (\eth, \in)^s \text{ or} \\ (0, 0) \in (\hbar, q)_t \times (\eth, q)^s \end{array} \right), \tag{3.13}$$

$$\begin{array}{l} \left(\begin{array}{l} (x, \mathfrak{a}) \in (\hbar, \in)_{t_1} \times (\eth, \in)^{s_1} \\ (z^x|z^x, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \in (\hbar, \in)_{t_2} \times (\eth, \in)^{s_2} \end{array} \right) \\ \vdash \left(\begin{array}{l} (z, \mathfrak{c}) \in (\hbar, \in)_{\min\{t_1,t_2\}} \times (\eth, \in)^{\max\{s_1,s_2\}} \text{ or} \\ (z, \mathfrak{c}) \in (\hbar, q)_{\min\{t_1,t_2\}} \times (\eth, q)^{\max\{s_1,s_2\}} \end{array} \right), \end{array} \tag{3.14}$$

$$\hbar(0) \geq \min\{\hbar(x), 0.5\},\ \eth(0) \leq \max\{\eth(\mathfrak{a}), 0.5\}, \tag{3.15}$$

and

$$\left(\begin{array}{l} \hbar(z) \geq \min\{\hbar(x), \hbar(z^x|z^x), 0.5\} \\ \eth(\mathfrak{c}) \leq \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}), 0.5\} \end{array} \right) \tag{3.16}$$

for all $x, \mathfrak{a}, z, \mathfrak{c} \in X$ and $t, t_1, t_2, s, s_1, s_2 \in (0, 1)$. Then (3.13) and (3.14) are equivalent to (3.15) and (3.16), respectively.

Proof. Assume that (3.13) and (3.14) are established. If (3.15) is not valid, then $\hbar(0) < t \leq \min\{\hbar(x), 0.5\}$ or $\eth(0) > s \geq \max\{\eth(\mathfrak{a}), 0.5\}$ for some $x, \mathfrak{a} \in X$ and $(t,s) \in [0,0.5] \times [0.5,1]$. Thus $x \in (\hbar, \in)_t$ and $0 \notin (\hbar, \in)_t$, or $\mathfrak{a} \in (\eth, \in)^s$ and $0 \notin (\eth, \in)^s$. Also, $\hbar(0) + t < 1$ and $\eth(0) + s > 1$, i.e., $(0,0) \notin (\hbar, q)_t \times (\eth, q)^s$ which is a contradiction. For every $x, \mathfrak{a}, x, \mathfrak{c} \in X$, we put $t := \min\{\hbar(x), \hbar(z^x|z^x)\}$ and $s := \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})\}$. We consider the next two cases:

(i) $\left\{\begin{array}{l} t := \min\{\hbar(x), \hbar(z^x|z^x)\} < 0.5 \\ s := \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})\} > 0.5, \end{array}\right.$

(ii) $\left\{\begin{array}{l} t := \min\{\hbar(x), \hbar(z^x|z^x)\} \geq 0.5 \\ s := \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})\} \leq 0.5. \end{array}\right.$

The first case induces $(t,s) \in [0,0.5] \times [0.5,1]$, $(x, \mathfrak{a}) \in (\hbar, \in)_t \times (\eth, \in)^s$ and

$$(z^x|z^x, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \in (\hbar, \in)_t \times (\eth, \in)^s.$$

It follows from (3.14) that $(z, \mathfrak{c}) \in (\hbar, \in)_t \times (\eth, \in)^s$ or $(z, \mathfrak{c}) \in (\hbar, q)_t \times (\eth, q)^s$. If $(z, \mathfrak{c}) \in (\hbar, \in)_t \times (\eth, \in)^s$, then

$$\begin{aligned} \hbar(z) &\geq t = \min\{\hbar(x), \hbar(z^x|z^x)\} \geq \min\{\hbar(x), \hbar(z^x|z^x), 0.5\}, \\ \eth(\mathfrak{c}) &\leq s = \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})\} \leq \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}), 0.5\}. \end{aligned}$$

If $(z, \mathfrak{c}) \in (\hbar, q)_t \times (\eth, q)^s$, then

$$\begin{aligned} 1 &< \hbar(z) + t = \hbar(z) + \min\{\hbar(x), \hbar(z^x|z^x)\} < \hbar(z) + 0.5 \\ 1 &> \eth(\mathfrak{c}) + s = \eth(\mathfrak{c}) + \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})\} > \eth(\mathfrak{c}) + 0.5, \end{aligned}$$

which imply that $\hbar(z) > 1 - 0.5 = 0.5 \geq \min\{\hbar(x), \hbar(z^x|z^x), 0.5\}$ and

$$\eth(\mathfrak{c}) < 1 - 0.5 = 0.5 \leq \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}), 0.5\}.$$

For the second case, we have $(t,s) \in [0.5,1] \times [0,0.5]$, $(x, \mathfrak{a}) \in (\hbar, \in)_{0.5} \times (\eth, \in)^{0.5}$ and $(z^x|z^x, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \in (\hbar, \in)_{0.5} \times (\eth, \in)^{0.5}$. Then $(z, \mathfrak{c}) \in (\hbar, \in)_{0.5} \times (\eth, \in)^{0.5}$ or

$$(z, \mathfrak{c}) \in (\hbar, q)_{0.5} \times (\eth, q)^{0.5}$$

by (3.14). If $\hbar(z) < 0.5 < \eth(\mathfrak{c})$, then $\hbar(z) + 0.5 < 1 < \eth(\mathfrak{c}) + 0.5$, i.e., $(z, \mathfrak{c}) \notin (\hbar, q)_{0.5} \times (\eth, q)^{0.5}$. Thus $(z, \mathfrak{c}) \in (\hbar, \in)_{0.5} \times (\eth, \in)^{0.5}$. So

$$\begin{aligned} \hbar(z) &\geq 0.5 \geq \min\{\hbar(x), \hbar(z^x|z^x), 0.5\} \\ \eth(\mathfrak{c}) &\leq 0.5 \leq \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}), 0.5\}. \end{aligned}$$

Conversely, suppose that (3.15) and (3.16) are established. Let $t, s \in (0,1)$ be such that $(\hbar, \in)_t \neq \varnothing \neq (\eth, \in)^s$, say $(x, \mathfrak{a}) \in (\hbar, \in)_t \times (\eth, \in)^s$. Then

$$\hbar(0) \geq \min\{\hbar(x), 0.5\} \geq \min\{t, 0.5\} = \left\{\begin{array}{ll} t & \text{if } t \leq 0.5 \\ 0.5 & \text{if } t > 0.5 \end{array}\right.$$

and

$$\eth(0) \leq \max\{\eth(\mathfrak{a}), 0.5\} \leq \max\{s, 0.5\} = \left\{\begin{array}{ll} s & \text{if } s \geq 0.5 \\ 0.5 & \text{if } s < 0.5. \end{array}\right.$$

Thus $(0,0) \in (\hbar, \in)_t \times (\eth, \in)^s$ for $(t,s) \in [0,0.5] \times [0.5,1]$. If $(t,s) \in (0.5,1] \times [0,0.5)$, then $\hbar(x) + t = 0.5 + t > 1$ and $\eth(\mathfrak{a}) + s = 0.5 + s < 1$, i.e., $(0,0) \in (\hbar, q)_t \times (\eth, q)^s$. So (3.13) is valid.

Let $x, \mathfrak{a}, z, \mathfrak{c} \in X$ and $t_1, t_2, s_1, s_2 \in (0,1)$ be such that $(x, \mathfrak{a}) \in (\hbar, \in)_{t_1} \times (\eth, \in)^{s_1}$ and $(z^x|z^x, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \in (\hbar, \in)_{t_2} \times (\eth, \in)^{s_2}$. Then $\hbar(x) \geq t_1$, $\eth(\mathfrak{a}) \leq s_1$, $\hbar(z^x|z^x) \geq t_2$ and $\eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \leq s_2$. It follows from (3.16) that

$$\begin{aligned}\hbar(z) &\geq \min\{\hbar(x), \hbar(z^x|z^x), 0.5\} \geq \min\{t_1, t_2, 0.5\} \\ &= \begin{cases} \min\{t_1, t_2\} & \text{if } \min\{t_1, t_2\} \leq 0.5 \\ 0.5 & \text{if } \min\{t_1, t_2\} > 0.5 \end{cases}\end{aligned}$$

and

$$\begin{aligned}\eth(\mathfrak{c}) &\leq \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}), 0.5\} \leq \max\{s_1, s_2, 0.5\} \\ &= \begin{cases} \max\{s_1, s_2\} & \text{if } \max\{s_1, s_2\} \geq 0.5 \\ 0.5 & \text{if } \max\{s_1, s_2\} < 0.5. \end{cases}\end{aligned}$$

If $(\min\{t_1, t_2\}, \max\{s_1, s_2\}) \in [0, 0.5] \times [0.5, 1]$, then

$$(z, \mathfrak{c}) \in (\hbar, \in)_{\min\{t_1,t_2\}} \times (\eth, \in)^{\max\{s_1,s_2\}}.$$

If $(\min\{t_1, t_2\}, \max\{s_1, s_2\}) \in (0.5, 1] \times [0, 0.5)$, then

$$\hbar(z) + \min\{t_1, t_2\} = 0.5 + \min\{t_1, t_2\} > 1$$

and

$$\eth(\mathfrak{c}) + \max\{s_1, s_2\} = 0.5 + \max\{s_1, s_2\} < 1,$$

that is, $(z, \mathfrak{c}) \in (\hbar, q)_{\min\{t_1,t_2\}} \times (\eth, q)^{\max\{s_1,s_2\}}$. Thus (3.14) is valid. □

Theorem 3.11. *Let $\mathcal{L} := (\hbar,\ \eth;\ \varepsilon,\ \delta)$ be a linear Diophantine set in X that satisfies (3.4) and $\hbar(x) < 0.5 < \eth(\mathfrak{a})$ for all $x, \mathfrak{a} \in X$. If it satisfies the conditions (3.13) and (3.14), then it is a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$.*

Proof. Let $t, s \in (0,1)$ be such that $(\hbar, \in)_t \neq \varnothing \neq (\eth, \in)^s$, say $(y, \mathfrak{b}) \in (\hbar, \in)_t \times (\eth, \in)^s$. Then $\hbar(y) \geq t$ and $\eth(\mathfrak{b}) \leq s$, which imply from Lemma 3.10 that

$$\hbar(0) \geq \min\{\hbar(y), 0.5\} = \hbar(y) \geq t$$

and $\eth(0) \leq \max\{\eth(\mathfrak{b}), 0.5\} = \eth(\mathfrak{b}) \leq s$. Thus $(0, 0) \in (\hbar, \in)_t \times (\eth, \in)^s$.

Let $x, \mathfrak{a}, z, \mathfrak{c} \in X$ and $t_1, t_2, s_1, s_2 \in (0,1)$ be such that $(x, \mathfrak{a}) \in (\hbar, \in)_{t_1} \times (\eth, \in)^{s_1}$ and $(z^x|z^x, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \in (\hbar, \in)_{t_2} \times (\eth, \in)^{s_2}$. Then $\hbar(x) \geq t_1$, $\eth(\mathfrak{a}) \leq s_1$, $\hbar(z^x|z^x) \geq t_2$, and $\eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \leq s_2$. It follows from Lemma 3.10 that

$$\hbar(z) \geq \min\{\hbar(x), \hbar(z^x|z^x), 0.5\} = \min\{\hbar(x), \hbar(z^x|z^x)\} \geq \min\{t_1, t_2\}$$

and

$$\eth(\mathfrak{c}) \leq \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}), 0.5\} = \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})\} \leq \max\{s_1, s_2\}.$$

Thus $(z, \mathfrak{c}) \in (\hbar, \in)_{\min\{t_1,t_2\}} \times (\eth, \in)^{\max\{s_1,s_2\}}$. So $\mathcal{L} := (\hbar,\ \eth;\ \varepsilon,\ \delta)$ is a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$. □

Theorem 3.12. *A linear Diophantine set $\mathcal{L} := (\hbar,\ \eth;\ \varepsilon,\ \delta)$ in X is a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$ if and only if the nonempty sets $\varepsilon_t := \{x \in X \mid \frac{1}{\varepsilon_x} \leq \frac{1}{t}\}$, $\delta^s := \{x \in X \mid \frac{1}{\delta_x} \geq \frac{1}{s}\}$, $(\hbar, \in)_t$ and $(\eth, \in)^s$ are ideals of $\mathcal{X}$ for all $(t, s) \in (0,1) \times (0,1)$.*

Proof. Let $\mathcal{L} := (\hbar,\ \eth;\ \varepsilon,\ \delta)$ be a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$. Let $(t,s) \in (0,1) \times (0,1)$ be such that $\varepsilon_t \neq \varnothing \neq \delta^s$, say $(x,\mathfrak{a}) \in \varepsilon_t \times \delta^s$. Using (3.4), we have $\frac{1}{\varepsilon_0} \leq \frac{1}{\min\{\varepsilon_x,\varepsilon_x\}} = \frac{1}{\varepsilon_x} \leq \frac{1}{t}$ and $\frac{1}{\delta_0} \geq \frac{1}{\max\{\delta_x,\delta_x\}} = \frac{1}{\delta_x} \geq \frac{1}{s}$. Then $(0,0) \in \varepsilon_t \times \delta^s$. Let $(x,\mathfrak{a}) \in \varepsilon_t \times \delta^s$ and $(z^x|z^x, \mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a}) \in \varepsilon_t \times \delta^s$ for every $x,\mathfrak{a},z,\mathfrak{c} \in X$. Then $\frac{1}{\varepsilon_x} \leq \frac{1}{t}$, $\frac{1}{\delta_\mathfrak{a}} \geq \frac{1}{s}$, $\frac{1}{\varepsilon_{z^x|z^x}} \leq \frac{1}{t}$, and $\frac{1}{\delta_{\mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a}}} \geq \frac{1}{s}$. It follows from (3.4) that $\frac{1}{\varepsilon_z} \leq \frac{1}{\min\{\varepsilon_x,\ \varepsilon_{z^x|z^x}\}} \leq \frac{1}{t}$ and $\frac{1}{\delta_\mathfrak{c}} \geq \frac{1}{\max\{\delta_\mathfrak{a},\ \delta_{\mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a}}\}} \geq \frac{1}{s}$. Thus $(z,\mathfrak{c}) \in \varepsilon_t \times \delta^s$. This shows that ε_t and δ^s are ideals of $\mathcal{X}$. Suppose $(\hbar,\in)_t$ and $(\eth,\in)^s$ are nonempty. Then $(0,0) \in (\hbar,\in)_t \times (\eth,\in)^s$ by (3.5). Let $(x,\mathfrak{a}) \in (\hbar,\in)_t \times (\eth,\in)^s$ and $(z^x|z^x, \mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a}) \in (\hbar,\in)_t \times (\eth,\in)^s$ for every $x,\mathfrak{a},z,\mathfrak{c} \in X$. Then $(z,\mathfrak{c}) \in (\hbar,\in)_t \times (\eth,\in)^s$ by (3.6). Thus $(\hbar,\in)_t$ and $(\eth,\in)^s$ are ideals of $\mathcal{X}$.

Conversely, suppose that ε_t and δ^s are nonempty ideals of $\mathcal{X}$ for all $(t,s) \in (0,1) \times (0,1)$. If $((0,x,x),(0,\mathfrak{a},\mathfrak{a})) \notin \varepsilon(\min) \times \delta(\max)$ for some $x,\mathfrak{a} \in X$, then $\frac{1}{\varepsilon_0} > \frac{1}{\min\{\varepsilon_x,\varepsilon_x\}} = \frac{1}{\varepsilon_x}$ or $\frac{1}{\delta_0} < \frac{1}{\max\{\delta_\mathfrak{a},\delta_\mathfrak{a}\}} = \frac{1}{\delta_\mathfrak{a}}$. Thus $0 \notin \varepsilon_t$ or $0 \notin \delta^s$, which is a contradiction. So

$$((0,x,x),(0,\mathfrak{a},\mathfrak{a})) \in \varepsilon(\min) \times \delta(\max)$$

for all $x,\mathfrak{a} \in X$. For every $x,\mathfrak{a},z,\mathfrak{c} \in X$, if we take $t = \min\{\varepsilon_x, \varepsilon_{z^x|z^x}\}$ and $s = \max\{\delta_\mathfrak{a}, \delta_{\mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a}}\}$, then $\frac{1}{\varepsilon_x} \leq \frac{1}{\min\{\varepsilon_x,\,\varepsilon_{z^x|z^x}\}} = \frac{1}{t}$, $\frac{1}{\varepsilon_{z^x|z^x}} \leq \frac{1}{\min\{\varepsilon_x,\,\varepsilon_{z^x|z^x}\}} = \frac{1}{t}$, $\frac{1}{\delta_\mathfrak{a}} \geq \frac{1}{\max\{\delta_\mathfrak{a},\,\delta_{\mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a}}\}} = \frac{1}{s}$, and $\frac{1}{\delta_{\mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a}}} \geq \frac{1}{\max\{\delta_\mathfrak{a},\,\delta_{\mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a}}\}} = \frac{1}{s}$. Thus $(x,\mathfrak{a}) \in \varepsilon_t \times \delta_s$ and $(z^x|z^x, \mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a}) \in \varepsilon_t \times \delta_s$. So $(z,\mathfrak{c}) \in \varepsilon_t \times \delta_s$. It follows that $\frac{1}{\varepsilon_z} \leq \frac{1}{t} = \frac{1}{\min\{\varepsilon_x,\varepsilon_{z^x|z^x}\}}$ and $\frac{1}{\delta_\mathfrak{c}} \geq \frac{1}{s} = \frac{1}{\max\{\delta_\mathfrak{a},\delta_{\mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a}}\}}$. Hence

$$((z,x,z^x|z^x),(\mathfrak{c},\mathfrak{a},\mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a})) \in \varepsilon(\min) \times \delta(\max)$$

for all $x,\mathfrak{a},z,\mathfrak{c} \in X$. It is obvious that (3.5) and (3.6) are established if $(\hbar,\in)_t$ and $(\eth,\in)^s$ are nonempty ideals of $\mathcal{X}$ for all $(t,s) \in (0,1) \times (0,1)$. Consequently, $\mathcal{L} := (\hbar,\ \eth;\ \varepsilon,\ \delta)$ is a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$. $\square$

Theorem 3.13. *If a linear Diophantine set $\mathcal{L} := (\hbar,\ \eth;\ \varepsilon,\ \delta)$ in X satisfies*

$$(\forall x,\mathfrak{a} \in X)\left(\frac{1}{\varepsilon_x} \geq \frac{1}{\max\{\varepsilon_0, 0.5\}},\ \frac{1}{\delta_\mathfrak{a}} \leq \frac{1}{\min\{\delta_0, 0.5\}}\right), \tag{3.17}$$

$$(\forall x,\mathfrak{a},z,\mathfrak{c} \in X)\left(\begin{array}{l} \frac{1}{\min\{\varepsilon_x,\varepsilon_{z^x|z^x}\}} \geq \frac{1}{\max\{\varepsilon_z, 0.5\}} \\ \frac{1}{\max\{\delta_\mathfrak{a},\ \delta_{\mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a}}\}} \leq \frac{1}{\min\{\varepsilon_\mathfrak{c}, 0.5\}} \end{array}\right), \tag{3.18}$$

$$(\forall x,\mathfrak{a} \in X)\left(\begin{array}{l} \hbar(x) \leq \max\{\hbar(0),\ 0.5\} \\ \eth(\mathfrak{a}) \geq \min\{\eth(0),\ 0.5\} \end{array}\right), \tag{3.19}$$

$$\begin{array}{l}(\forall x,\mathfrak{a},z,\mathfrak{c} \in X) \\ \left(\begin{array}{l} \min\{\hbar(x),\ \hbar(z^x|z^x)\} \leq \max\{\hbar(z),\ 0.5\} \\ \max\{\eth(\mathfrak{a}),\ \eth(\mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a})\} \geq \min\{\eth(\mathfrak{c}),\ 0.5\} \end{array}\right),\end{array} \tag{3.20}$$

then the nonempty sets ε_t, δ^s, $(\hbar,\in)_t$ and $(\eth,\in)^s$ are ideals of $\mathcal{X}$ for all $(t,s) \in (0.5,1) \times (0,0.5)$.

Proof. Let $(x,\mathfrak{a}) \in \varepsilon_t \times \delta^s$ and $(z^x|z^x, \mathfrak{c}^\mathfrak{a}|\mathfrak{c}^\mathfrak{a}) \in \varepsilon_t \times \delta^s$ for $(t,s) \in (0.5,1) \times (0,0.5)$. Using (3.17) induces $\max\{\varepsilon_0,\ 0.5\} \geq \varepsilon_x \geq t > 0.5$ and $\min\{\delta_0,\ 0.5\} \leq \delta_\mathfrak{a} \leq s < 0.5$. Then $\varepsilon_0 \geq t$ and $\delta_0 \leq s$. Thus $\frac{1}{\varepsilon_0} \leq \frac{1}{t}$ and $\frac{1}{\delta_0} \geq \frac{1}{s}$, which shows that $(0,0) \in \varepsilon_t \times \delta^s$. Also, using (3.18), we have $\max\{\varepsilon_z,\ 0.5\} \geq \min\{\varepsilon_x,\ \varepsilon_{z^x|z^x}\} \geq \min\{t,t\} = t > 0.5$

and $\min\{\delta_{\mathfrak{c}},\, 0.5\} \leq \max\{\delta_{\mathfrak{a}},\, \delta_{\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}}\} \leq \max\{s, s\} = s < 0.5$. So $\varepsilon_z \geq t$ and $\delta_{\mathfrak{c}} \leq s$, which imply that $\frac{1}{\varepsilon_z} \leq \frac{1}{t}$ and $\frac{1}{\delta_{\mathfrak{c}}} \geq \frac{1}{s}$. Hence $(z, \mathfrak{c}) \in \varepsilon_t \times \delta^s$. Therefore ε_t and δ^s are ideals of $\mathcal{X}$. Let $(x, \mathfrak{a}) \in (\hbar, \in)_t \times (\eth, \in)^s$ and $(z^x|z^x, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \in (\hbar, \in)_t \times (\eth, \in)^s$ for $(t, s) \in (0.5, 1) \times (0, 0.5)$. Then $\hbar(x) \geq t > 0.5$, $\eth(\mathfrak{a}) \leq s < 0.5$, $\hbar(z^x|z^x) \geq t > 0.5$, and $\eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \leq s < 0.5$. It follows from (3.19) and (3.20) that $\max\{\hbar(0),\, 0.5\} \geq \hbar(x) \geq t > 0.5$, $\min\{\eth(0),\, 0.5\} \leq \eth(\mathfrak{a}) \leq s < 0.5$ and

$$\begin{aligned}
&\max\{\hbar(z),\, 0.5\} \geq \min\{\hbar(x),\, \hbar(z^x|z^x)\} \geq t > 0.5,\\
&\min\{\eth(\mathfrak{c}),\, 0.5\} \leq \max\{\eth(\mathfrak{a}),\, \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})\} \leq s < 0.5.
\end{aligned}$$

Thus $\hbar(0) \geq t$ and $\eth(0) \leq s$, that is, $(0, 0) \in (\hbar, \in)_t \times (\eth, \in)^s$, and $\hbar(z) \geq t$ and $\eth(\mathfrak{c}) \leq s$, that is, $(z, \mathfrak{c}) \in (\hbar, \in)_t \times (\eth, \in)^s$. So $(\hbar, \in)_t$ and $(\eth, \in)^s$ are ideals of $\mathcal{X}$. □

Theorem 3.14. *If $\mathcal{L} := (\hbar,\, \eth;\, \varepsilon,\, \delta)$ is a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$, then the nonempty sets*

$$\varepsilon_{q(t)} := \{x \in X \mid \tfrac{1}{\varepsilon_x} < \tfrac{1}{1-t}\},\ \delta^{q(s)} := \{x \in X \mid \tfrac{1}{\delta_x} > \tfrac{1}{1-s}\},$$

$(\hbar, q)_t$ and $(\eth, q)^s$ are ideals of $\mathcal{X}$ for all $(t, s) \in (0, 1) \times (0, 1)$.

Proof. Assume that $\mathcal{L} := (\hbar,\, \eth;\, \varepsilon,\, \delta)$ is a linear Diophantine Sheffer stroke ideal of $\mathcal{X}$. Let $x, \mathfrak{a}, z, \mathfrak{c} \in X$ and $(t, s) \in (0, 1) \times (0, 1)$ be such that $(x,\, \mathfrak{a}) \in \varepsilon_{q(t)} \times \delta^{q(s)}$ and $(z^x|z^x,\, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \in \varepsilon_{q(t)} \times \delta^{q(s)}$. Then

$$\begin{aligned}
&\tfrac{1}{\varepsilon_0} \leq \tfrac{1}{\min\{\varepsilon_x,\, \varepsilon_x\}} = \tfrac{1}{\varepsilon_x} < \tfrac{1}{1-t},\\
&\tfrac{1}{\delta_0} \geq \tfrac{1}{\max\{\delta_{\mathfrak{a}},\, \delta_{\mathfrak{a}}\}} = \tfrac{1}{\delta_{\mathfrak{a}}} > \tfrac{1}{1-s},\\
&\tfrac{1}{\varepsilon_z} \leq \tfrac{1}{\min\{\varepsilon_x,\, \varepsilon_{z^x|z^x}\}} < \tfrac{1}{1-t},\\
&\tfrac{1}{\delta_{\mathfrak{c}}} \geq \tfrac{1}{\max\{\delta_{\mathfrak{a}},\, \delta_{\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}}\}} > \tfrac{1}{1-s}
\end{aligned}$$

by (3.4). Thus $(0, 0) \in \varepsilon_{q(t)} \times \delta^{q(s)}$ and $(z, \mathfrak{c}) \in \varepsilon_{q(t)} \times \delta^{q(s)}$. This shows that $\varepsilon_{q(t)}$ and $\delta^{q(s)}$ are ideals of $\mathcal{X}$. Let $(x,\, \mathfrak{a}) \in (\hbar, q)_t \times (\eth, q)^s$ and $(z^x|z^x,\, \mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) \in (\hbar, q)_t \times (\eth, q)^s$ for every $x, \mathfrak{a}, z, \mathfrak{c} \in X$ and $(t, s) \in (0, 1) \times (0, 1)$. Then $\hbar(x) + t > 1$, $\eth(\mathfrak{a}) + s < 1$, $\hbar(z^x|z^x) + t > 1$, and $\eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) + s < 1$. It follows from (3.8) and (3.9) that $\hbar(0) + t \geq \hbar(x) + t > 1$, $\eth(0) + s \leq \eth(\mathfrak{a}) + s < 1$, that is, $(0,\, 0) \in (\hbar, q)_t \times (\eth, q)^s$, and

$$\begin{aligned}
\hbar(z) + t &\geq \min\{\hbar(x), \hbar(z^x|z^x)\} + t\\
&= \min\{\hbar(x) + t, \hbar(z^x|z^x) + t\} > 1,
\end{aligned}$$

$$\begin{aligned}
\eth(\mathfrak{c}) + s &\leq \max\{\eth(\mathfrak{a}), \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}})\} + s\\
&= \max\{\eth(\mathfrak{a}) + s, \eth(\mathfrak{c}^{\mathfrak{a}}|\mathfrak{c}^{\mathfrak{a}}) + s\} < 1,
\end{aligned}$$

that is, $(z,\, \mathfrak{c}) \in (\hbar, q)_t \times (\eth, q)^s$. Thus $(\hbar, q)_t$ and $(\eth, q)^s$ are ideals of $\mathcal{X}$. □

Acknowledgment

The author would like to thank the anonymous reviewers for their valuable suggestions.

References

[1] H. M. Sheffer, A set of five independent postulates for Boolean algebras, Transactions of the American Mathematical Society 14 (4) (1913), 481–488. https://doi.org/10.2307/1988701

[2] T. Oner, T. Katican and A. Borumand Saeid, Relation between Sheffer stroke and Hilbert Algebras, Categories and General Algebraic Structures with Applications 14 (1) (2021) 245–268. https://doi.org/10.29252/cgasa.14.1.245

[3] T. Katican and H. Bordbar, Sheffer stroke Hilbert algebras stabilizing by ideals, Axioms 2024, 13, 97. https://doi.org/10.3390/axioms13020097

[4] Y. B. Jun and T. Oner, Weak filters and multipliers in Sheffer stroke Hilbert algebras, Palestine Journal of Mathematics 14 (2) (2025) 749–761.

[5] Y. B. Jun and T. Oner, Ideals of Sheffer stroke Hilbert algebras based on fuzzy points, Honam Mathematical Journal 46 (1) (2024) 82–100. https://doi.org/10.5831/HMJ.2024.46.1.82

[6] T. Oner, T. Katican and A. Borumand Saeid, Fuzzy ideals of Sheffer stroke Hilbert algebras, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences 93 (2023), 85–94. https://doi.org/10.1007/s40010-022-00794-9

[7] M. Riaz and M. R. Hashmi, Linear Diophantine fuzzy set and its applications towards multi-attribute decision making problems, Journal of Intelligent & Fuzzy Systems 37 (4) (2019) 5417–5439. https://doi.org/10.3233/JIFS-190550

Y. B. Jun (skywine@gmail.com)
Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea
ORCID iD: https://orcid.org/0000-0002-0181-8969